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Dynamical Exponents for One-Dimensional Random–Random Directed Walks

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The dynamical exponents of the coordinate and of the mean square displacement are explicitly calculated in the case of a directed random walk on a one-dimensional random lattice. Moreover, it is shown that, in the dynamical phase where the coordinate increases slower than t, the latter is not a self-averaging quantity.

KEY WORDS: Brownian motion; random walks; disordered media.

1. INTRODUCTION AND BASIC EQUATIONS

We consider the one-dimensional random directed walk on a disordered lattice described by the following master equation:

$$\frac{dp_n}{dt} = -W_n p_n + W_{n-1} p_{n-1} \tag{1}$$

where $p_n(t)$ denotes the probability to be at the site of label n at time t. The W's are nonnegative quantities chosen independently at random in a given probability distribution $\rho(W)$.

Equation (1) is the directed version of the master equation traditionally used in models of transport in biased media with *quenched* disorder (see, for instance, ref. 1 for the symmetric case and refs. 2 and 3 for recent

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reviews on asymmetric models). Note that it can also describe a random process in which the variable of interest can only increase, starting from a given initial value. Thus, for instance, the dynamics of the number of charged particles deposited at random on an electrode can be modeled by Eq. (1).[4] Furthermore, the general both-way asymmetric walk is believed to be asymptotically similar to a directed walk on a renormalized lattice[5,6]; in this respect, the study of the directed walk is interesting in that it can generate the basic features of the general problem in a simplified framework. In particular, it allows one to characterize the dynamical phases introduced in ref. 7 by, as shown below, explicitly calculating the appropriate exponents. Moreover, we are able to demonstrate that, when the coordinate increases slower than t, it still fluctuates in the final regime; otherwise stated, it is not a self-averaging quantity.

The main goal is to qualify at best the dynamical regime at large times, that is, to find the asymptotic behavior of the quantities

$$\overline{x(t)} = \sum_{n=0}^{+\infty} n p_n(t) \tag{2}$$

$$\overline{x^2(t)} = \sum_{n=0}^{+\infty} n^2 p_n(t) \tag{3}$$

Clearly, Eq. (1) can be viewed as describing the motion of a particle on a one-dimensional lattice; then, $\overline{x(t)}$ and $\overline{x^2(t)}$ are, respectively, the coordinate and the mean square displacement of this particle; the lattice spacing is taken as unity and $n=0$ denotes the initial position. In all the following, an overbar means an expectation value (usually called "thermal average") computed with the p's, still *a priori* depending on the particular sampling of the W's. On the other hand, $\langle \cdots \rangle$ stands for a disorder average taken through the use of $\rho(W)$.

The nature of $\rho(W)$ basically determines the characteristics of the final dynamics. By an exact resummation of the series appearing in Eqs. (2) and (3), it has been shown[8] that, when all the inverse moments $M_{-n} = \langle W^{-n} \rangle$ are bounded, a standard regime exists with finite velocity V and diffusion coefficient D. Furthermore, it was proved there that both V and D are self-averaging quantities, that is, are indeed independent of the sample considered.

Here we aim at analyzing the very different situation where, on the contrary, the probability is high to have a nearly broken link. This will be modeled by taking

$$\rho(W) = C_\mu W^{\mu-1} f_C\left(\frac{W}{W_m}\right) \qquad (W > 0,\ \mu > 0) \tag{4}$$

C_μ is the normalization constant, μ a strictly positive number, and f_C a cutoff function basically specified by the fixed frequency W_m. Note that the smaller is μ, the higher is the probability to find a quasi-broken link. One thus expects a slowing down of the motion when μ is decreased toward 0_+.

We shall here only compute disorder-averaged quantities, since, in that case, it seems impossible to generalize the successful resummation given in ref. 8. It will nevertheless be possible to show that for $\mu < 1$, $\overline{x(t)}$ still exhibits strong sample-to-sample fluctuations, even at large times.

Equation (1) can be solved by a Laplace transformation. Denoting by $P_n(z)$ the Laplace transform of $p_n(t)$, we can rewrite Eq. (1) as

$$zP_n(z) - \delta_{n,0} = -W_n P_n(z) + W_{n-1} P_{n-1}(z) \tag{5}$$

yielding

$$P_0(z) = \frac{1}{z + W_0}, \qquad P_{n>0}(z) = P_0(z) \prod_{m=1}^{n} \frac{W_{m-1}}{z + W_m} \tag{6}$$

Introducing the generating function $\Phi(\phi, z)$,

$$\Phi(\phi, z) = \sum_{n=0}^{+\infty} e^{in\phi} P_n(z) \tag{7}$$

we see that the disorder average of Φ is equal to

$$\langle \Phi(\phi, z) \rangle = \frac{R(z)}{1 - e^{i\phi} + z e^{i\phi} R(z)} \tag{8}$$

where $R(z)$ is the Stieltjes transform of $\rho(W)$ defined as

$$R(z) = \int_0^{+\infty} dW \frac{\rho(W)}{z + W} \tag{9}$$

Note that when $n_0 < \mu \leqslant n_0 + 1$, where n_0 is a nonnegative integer, the first n_0 inverse moments of ρ exist and appear as coefficients of the integer powers of z in the nonsingular part of the expansion of $R(z)$ near $z = 0$. Calling $x_1(z)$ and $x_2(z)$ the Laplace transforms of $\overline{x(t)}$ and $\overline{x^2(t)}$ and differentiating twice Φ at $\phi = 0$, we find for the disorder averages

$$\langle x_1(z) \rangle = \frac{1}{z^2 R(z)} - \frac{1}{z} \tag{10}$$

$$\langle x_2(z) \rangle = \frac{2}{z^3 R^2(z)} - \frac{3}{z^2 R(z)} + \frac{1}{z} \tag{11}$$

The disorder-averaged mean square dispersion $\langle \overline{\Delta x^2(t)} \rangle$ is

$$\langle \overline{\Delta x^2(t)} \rangle = \langle \overline{x^2(t)} - \overline{x(t)}^2 \rangle \tag{12}$$

and can be obtained by a Laplace inversion of the quantity $\Delta_2(z)$ defined as

$$\langle \Delta_2(z) \rangle = \langle x_2(z) - (x_1 * x_1)(z) \rangle \tag{13}$$

In the latter equation, $F * G$ denotes the convolution

$$(F * G)(z) = \int_C \frac{dz'}{2i\pi} F(z')\, G(z - z')$$

In order to get a closed expression for the average $\langle x_1 * x_1 \rangle$, we note the functional relation valid for any given sampling of the lattice:

$$z x_1(z; W_0, W_1, W_2, \ldots) = \frac{W_0}{z + W_0} [1 + z x_1(z; W_1, W_2, W_3, \ldots)] \tag{14}$$

By applying this relation twice, using the fact that the W's are dummy variables when the disorder average is taken and that they are uncorrelated, we find

$$\langle x_1(z)\, x_1(z') \rangle = \frac{z - z'}{zz'} \left[1 - \frac{z^2 R(z) - z'^2 R(z')}{z - z'} \right] \frac{1/zR(z) + 1/z'R(z') - 1}{z^2 R(z) - z'^2 R(z')} \tag{15}$$

The expression (15) is the central quantity to be analyzed since, combined with $\langle x_2(z) \rangle$, it eventually leads to the mean square displacement. Before tackling this point, let us make precise the dynamical behavior of the simpler quantity $\langle \overline{x(t)} \rangle$.

2. PROBABILITY OF RETURN TO THE ORIGIN AND DYNAMICAL EXPONENTS FOR $\langle \overline{x(t)} \rangle$

The dynamics at large times is determined by the properties of the function $R(z)$ near $z = 0$. From its very definition, the latter can have singularities only on the half-line $\operatorname{Re} z < 0$. More precisely, this function has a cut on the negative real axis, going from $-W_{\min}$ to $-W_{\max}$ if $\rho(W)$ does not identically vanish in the interval $[W_{\min}, W_{\max}]$. For instance, by taking $f_C(x)$ as the unit step function $\theta(1 - x)$ in Eq. (4), the cut extends from zero to $-W_m$. With the same sharp cutoff function at $W = W_m$, $R(z)$ can be explicitly computed and, according to the value of μ, has the following expansions:

$$R(z)=\frac{\mu}{W_m}\left[\frac{\pi}{\sin\pi\mu}Z^{\mu-1}-\sum_{n=0}^{+\infty}\frac{(-Z)^n}{n+1-\mu}\right],\qquad 0<\mu<1 \tag{16}$$

$$R(z)=\frac{\mu}{W_m}\left[\sum_{n=1}^{n_0}\frac{(-Z)^{n-1}}{\mu-n}+\frac{\pi}{\sin\pi\mu}Z^{\mu-1}-\sum_{n=n_0}^{+\infty}\frac{(-Z)^n}{n+1-\mu}\right],$$

$$1\leqslant n_0<\mu\leqslant n_0+1 \tag{17}$$

In the above, Z denotes z/W_m and n_0 is a positive integer. When μ approaches such an integer, the appropriate limit has to be taken and the multivalued terms z^μ in the above produce logarithms.

The disorder-averaged probability of return to the origin $\langle P_0(z)\rangle$ is equal to $R(z)$ [see Eq. (6)]. Using the fact that $R(z)$ has no singularity except for its cut, the Laplace inversion produces the following expression:

$$\langle p_0(t)\rangle=\int_0^{+\infty}dW\,e^{-Wt}\rho(W) \tag{18}$$

which in turn yields

$$\langle p_0(t)\rangle=\frac{\mu}{(W_m t)^\mu}\gamma(\mu, W_m t) \tag{19}$$

where γ denotes the incomplete gamma function. The asymptotic behavior of $\langle p_0(t)\rangle$ is

$$\langle p_0(t)\rangle\approx\frac{\Gamma(\mu+1)}{(W_m t)^\mu} \tag{20}$$

$p_0(t)$ provides a good example of the fact that the disorder average can modify the time dependence. Indeed, from Eq. (6), it is readily seen that, for a given sample, $p_0(t)$ is exactly given at any time by $\exp(-W_0 t)$. Thus, averaging over disorder transforms this exponential into a powerlike long-time tail (at large times). In addition, it is worth noting that, in contrast to the exponents for the coordinate and the mean square dispersion derived below, the exponent for $\langle p_0(t)\rangle$ has the same functional dependence on μ for any μ. The quantity $\langle p_0(t)\rangle$ has been computed for the general walk[2,3,9] and displays the same behavior as $\langle p_0(t)\rangle$ given by Eq. (20).

Using Eq. (10), we find that the disorder-averaged coordinate has the following asymptotic behaviors ($T = W_m t$):

$$\langle \overline{x(t)} \rangle \approx \frac{\sin \pi\mu}{\pi\mu\Gamma(\mu+1)} T^{\mu}, \qquad 0 < \mu < 1 \tag{21}$$

$$\langle \overline{x(t)} \rangle \approx \frac{\mu-1}{\mu} T, \qquad 1 < \mu \tag{22}$$

The results expressed by Eqs. (21) and (22) are in agreement with ref. 3, obtained independently by another method. As expected,[3,8] for $\mu > 1$, a finite velocity exists and is given by

$$V = \frac{\mu - 1}{\mu} \equiv \frac{1}{M_{-1}} \tag{23}$$

On the contrary, for $\mu < 1$, the coordinate increases slower than t due to the greater weight of quasi broken links. It will be shown below that, in addition, the thermal average (2) is not in this case a self-averaging quantity, as a result of strong disorder. It may be understood as a consequence of the slowing down of the motion which prevents the particle from having a good feeling of the surrounding disorder. For $\mu = 1$, $\langle \overline{x(t)} \rangle$ behaves like $t/\ln t$.

3. DYNAMICAL EXPONENTS FOR $\langle \Delta x^2(t) \rangle$

The analysis of the mean square dispersion is, as usual, much more involved, since one has to obtain the small-z behavior of the convolution integral. After some algebra and using contour integration, the expression for $\langle \Delta_2(z) \rangle$ can be cast in the form

$$\langle \Delta_2(z) \rangle = \int_0^{+\infty} dx \, \frac{x(2x+z)}{x+z} \frac{\rho(x)}{A} \left[\frac{2}{(x+z)\, R(x+z)} - 1 \right] + \text{Res} \tag{24}$$

where A denotes the quantity

$$A = [x^2 R(xe^{-i\pi}) - (x+z)^2 R(x+z)][x^2 R(xe^{+i\pi}) - (x+z)^2 R(x+z)] \tag{25}$$

In Eq. (24) the first term originates from the cut of the multivalued function $R(z)$, whereas Res notes the contribution from the residues due to the zeros of the denominator in Eq. (15). For clarity, we now analyze separately the different cases.

3.1. $0<\mu<1$

In this case, the dominant contribution to the expression (25) comes from the lace integral (in fact, in this case, no pole merges to the origin in the limit $z \to 0$) and it turns out that

$$\langle \overline{\Delta x^2(t)} \rangle \approx \frac{1}{\Gamma(2\mu)} \left(\frac{\sin \pi\mu}{\pi\mu} \right)^3 I(\mu)(W_m t)^{2\mu} \tag{26}$$

where Γ denotes the Euler function and where $I(\mu)$ is the integral

$$I(\mu) = \int_0^{+\infty} dx \frac{2x+1}{(x+1)^{\mu+1}} \frac{x^\mu}{x^{2\mu+2} + 2\cos \pi\mu \, x^{\mu+1}(1+x)^{\mu+1} + (1+x)^{2\mu+2}}$$

The main result is the time exponent characterizing the asymptotic regime and equal to 2μ. This clearly indicates a slowing down of the motion at small μ, the more since the coefficient in (26) goes to zero like μ (see Fig. 1). Up to $\mu = \frac{1}{2}$, the motion is subdiffusive: because of the many nearly broken links, the spreading of the packet is very slow, as well as its

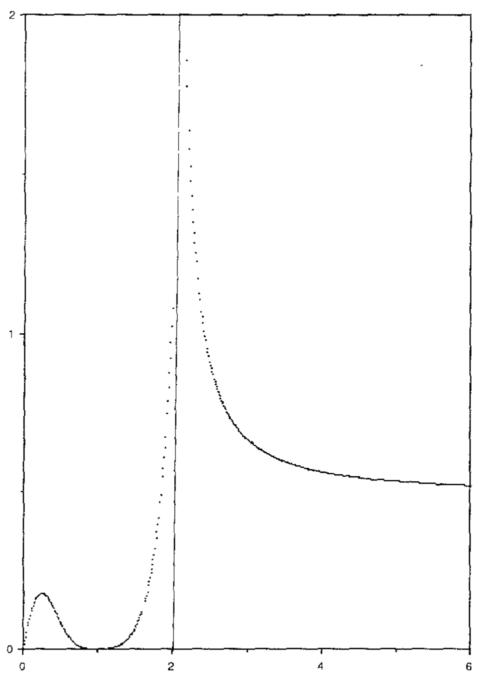

Fig. 1. Variations as a function of μ of the coefficient of $(W_m t)^{\alpha(\mu)}$ giving the dominant contribution to the mean square dispersion at large times [see Eqs. (26)–(28)].

translational motion. On the contrary, for $\frac{1}{2}<\mu<1$, such links are less frequently encountered and the motion speeds up a little; nevertheless, the particle may happen to be trapped and this produces a dragging effect displayed by a superdiffusive regime.

3.2. $1<\mu<2$

Again, the dominant contribution arises from the cut integral. A straightforward but tedious calculation leads to

$$\langle\overline{\Delta x^2(t)}\rangle \approx \frac{(\mu-1)^3\,\Gamma(\mu)}{\mu(3-\mu)(2-\mu)}\,F(1,\,\mu+1;3,\,-1)(W_m t)^{3-\mu} \tag{27}$$

where F is the hypergeometric function. Again, the regime is superdiffusive, but the exponent now decreases from 2 to 1 when μ increases from 1 to 2, as a result of the less and less frequently encountered quasibroken links. The global coefficient can be numerically computed and is plotted in the middle part of Fig. 1. The divergence at $\mu=2$ signals the onset of a non-analytical behavior. Actually, for $\mu=2$, the mean square dispersion increases as $t\ln t$. Using general statistical arguments, it is shown in ref. 3 that, for a discrete time model, Levy's laws are involved in the problem; an estimation of the behavior of the spreading also leads to a time dependence dominated by a $t^{3-\mu}$ term. We find, moreover, that the subdominant contribution (arising from the residue) is governed by a $t^{2/\mu}$ term.

3.3. $\mu>2$

Now, the two first inverse moments exist and, as expected, the regime is a standard drift-diffusion one. One finds

$$\langle\overline{\Delta x^2(t)}\rangle \approx 2\,\frac{M_{-2}}{2M_{-1}^3}\,t = 2\,\frac{(\mu-1)^3}{2\mu^2(\mu-2)}\,W_m t \tag{28}$$

It is interesting to note that for $\mu>2$, the dominant contribution to the rhs of Eq. (24) now comes from the pole, which behaves as $Z^{1/2}$ when $Z\ll 1$. The diffusion constant D is plotted in the right part of Fig. 1. Note that, when μ becomes very large, D, as expressed in units W_m, approaches its final value $(=\frac{1}{2})$ from below. This simply means that the diffusion is slower in the disordered lattice as compared to the ordered one characterized by W_m, not that weak disorder slows down diffusion. Were the ordered lattice used as reference characterized by a transfer rate equal to $1/M_{-1}$, one would find that D as expressed in this latter unit is greater for the disordered lattice than for the pure one.

As a summary, the dominant exponent is plotted in Fig. 2 as a function of μ. The interplay between dragging and slowing down is displayed by the nonmonotonic variation for $\mu < 2$. Roughly speaking, for $\mu < 1$, the numerous broken links slow down the motion of the center of the packet. For $\mu < \frac{1}{2}$, they are even so numerous as to hinder the spreading. For $\frac{1}{2} < \mu < 1$, they become less efficient: the center speeds up while the spreading becomes superdiffusive. For $1 < \mu < 2$, the spreading is still superdiffusive, while the number of quasibroken links is not large enough to forbid ordinary drift. Finally, for $\mu > 2$, these links are so rare that the standard drift-diffusion regime is restored.

4. NON-SELF-AVERAGING PROPERTY OF $\overline{x(t)}$ FOR $\mu < 1$

The sample-to-sample fluctuations of $\overline{x(t)}$ can be displayed by analyzing the relative dispersion linked to disorder, δ:

$$\delta = \frac{\langle [\overline{x(t)}]^2 \rangle - [\langle \overline{x(t)} \rangle]^2}{[\langle \overline{x(t)} \rangle]^2} \tag{29}$$

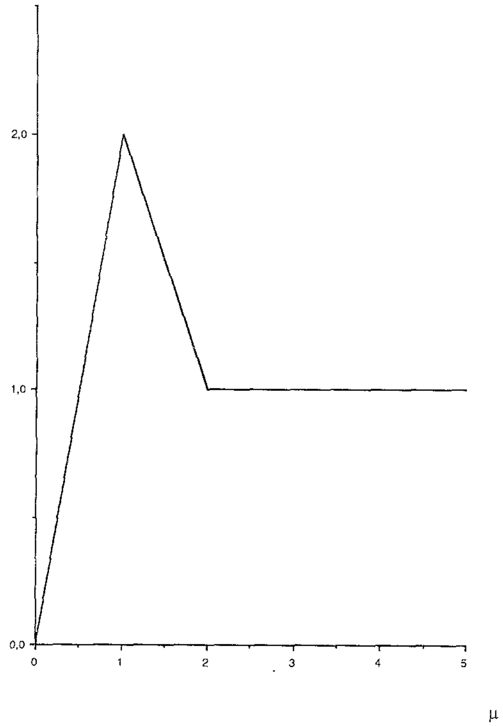

Fig. 2. Variations as a function of the dynamical exponent $\alpha(\mu)$ characterizing the dominant term in $\langle \overline{\Delta x^2(t)} \rangle$. For $\mu = 1, 2$, the behavior is not purely powerlike: for $\mu = 1$, $\langle \overline{\Delta x^2(t)} \rangle \propto t^2/\ln^3 t$; for $\mu = 2$, $\langle \overline{\Delta x^2(t)} \rangle \propto t \ln t$.

The convolution integral yields the first term in the numerator, while the second one is given by Eq. (21). These lead to

$$\delta = \frac{\mu}{2^{2\mu-1}} \frac{\Gamma(1/2)\,\Gamma(\mu)}{\Gamma(\mu+1/2)} - 1 \tag{30}$$

δ is a monotonically decreasing function for μ between 0 and 1, assuming its maximum value (equal to unity) for $\mu \to 0_+$ (see Fig. 3). This latter result establishes the fact that, due to the large disorder, the particle coordinate still fluctuates from one sample to another, even in the final dynamics. The self-averaging property is recovered for $\mu > 1$.

Clearly, our analysis cannot provide any information about the rather difficult problem of the behavior of the thermal average coordinate *for a given sample*. Indeed, in order to know the asymptotic time dependence (if it exists) for a definite configuration of the lattice, one should perform a similar study without taking any disorder average. Assuming for definiteness that, in any case, $\overline{x(t)}$ for a given sample behaves like At^{α} at large times, the following possibilities are open:

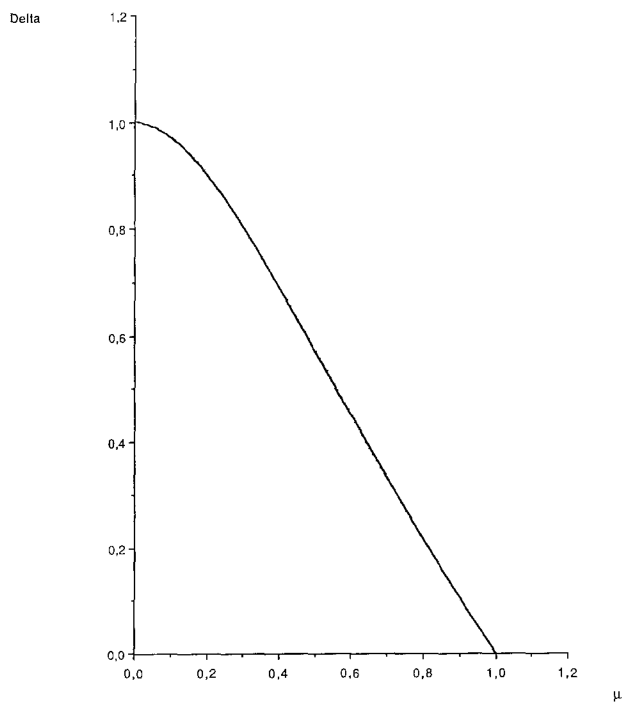

Fig. 3. Variations as a function of μ of the relative disorder fluctuation δ [see Eq. (30)].

(i) Only the prefactor A fluctuates from one sample to the other, while α is a fixed (self-averaging) exponent. Even in such a case, α could be different from μ; indeed, α could be equal to 1, for instance. Thus, $\overline{x(t)}$ would display the same phenomenon as $p_0(t)$ does: the time dependence, valid for any sample, is altered when an average over disorder is performed.

(ii) Both the prefactor A and the exponent α have sample-to-sample fluctuations. In this situation, no *a priori* characterization of a sample chosen at random would be possible.

The question of the underlying "microscopic" exponents yielding the dynamical ones here calculated for averaged quantities thus remains to be settled. We are presently investigating this point.

REFERENCES

1. S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, *Rev. Mod. Phys.* **53**:175 (1981).
2. J. P. Bouchaud, A. Comtet, A. Georges, and P. Le Doussal, *Ann. Phys.*, submitted.
3. J. P. Bouchaud and A. Georges, *Phys. Rep.*, in preparation.
4. C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, 1985).
5. J. Bernasconi and W. R. Schneider, in *Fractals in Physics*, L. Pietronero and E. Tosatti, eds. (Elsevier Science, 1986).
6. C. Aslangul, N. Pottier, and D. Saint-James, *J. Phys.* (Paris) **50**:899 (1989); and *J. Phys.* (Paris) **50**:1581 (1989) (erratum).
7. B. Derrida, *J. Stat. Phys.* **31**:433 (1983).
8. C. Aslangul, J. P. Bouchaud, A. Georges, N. Pottier, and D. Saint-James, *J. Stat. Phys.* **55**:1065 (1989).,
9. C. Aslangul, N. Pottier, and D. Saint-James, *Physica A*, to be published.